POTENTIALS OF POSITIVE MASS. PART II*

BY

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IV. The sweeping-out process

11. **Decreasing sequences of potentials.** As we have seen in §2.1, the limit of an increasing bounded sequence of potential functions of positive mass distributions on a bounded set F is itself a potential of positive mass. The limit of a decreasing sequence of such functions is however not necessarily superharmonic. Nevertheless, de la Vallée Poussin, in the memoir on the Poincaré sweeping-out process already cited in §1, is able to associate a positive mass distribution with a particular type of decreasing sequence, and the ideas underlying this association do not lose in force in a wider application.† Accordingly we shall consider an arbitrary monotone-decreasing sequence of potential functions of positive mass distributions on a bounded set F, which set without loss of generality may be assumed to be closed.

Let then $U_1, U_2, \cdots$ be a monotone-decreasing sequence of potentials

$$U_{i+1}(M) \leqq U_i(M), \qquad M \text{ in } W,$$

of positive mass distributions $f_1(e), f_2(e), \cdots$, respectively, on F. Denote the limiting function by $U_0(M)$. It is everywhere $\geqq 0$, but not necessarily superharmonic. It is harmonic in T.

The distributions $f_i(e)$ are bounded in their set, since, by §2, $f_i(F) \leqq f_1(F)$, $i > 1$, and accordingly the sequence contains a subsequence $\{f_{i_n}(e)\}$ which converges in the weak sense to a positive mass function $f(e)$ on F or on a subset of F, that is, converges so that

$$\lim_{i_n=\infty} \int_W \phi(M) df_{i_n}(e) = \int_W \phi(M) df(e)$$

for every continuous function $\phi(M)$. Let $U(M)$ be the potential of $f(e)$.

In particular,

$$(1) \qquad \int_W h^{1/p}(M, P) df(e_P) = \lim_{i_n=\infty} \int_W h^{1/p}(M, P) df_{i_n}(e_P).$$

* See these Transactions, vol. 37 (1935), pp. 226–253. Presented to the Society, December 29, 1932, and September 6, 1934; received by the editors January 14, 1935.

† These methods and ideas are closely related to those of N. Wiener and G. Bouligand. See G. Bouligand, *Fonctions Harmoniques. Principes de Picard et de Dirichlet*, Mémorial des Sciences Mathématiques, fascicule 11, Paris, 1926.

If M is distant δ from F, the equation (1) takes the form

$$U(M) = \lim_{i_n=\infty} U_{i_n}(M),$$

for an arbitrary value of ρ, $\rho < \delta$. Hence $U(M) = U_0(M)$, for M not on F.

If M is on F, we have from (1)

$$\int_W h^{1/\rho}(M, P)df(e_P) \leqq \liminf_{i_n=\infty} \int_W \frac{1}{MP} df_{i_n}(e_P) = \liminf_{i_n=\infty} U_{i_n}(M),$$

and since this relation is true for all ρ, we may let ρ approach zero and obtain the equation $U(M) \leqq U_0(M)$, M in F.

Finally, equation (1) is a statement of the fact that

$$A_U(\rho, M) = \lim_{i_n=\infty} A_{U_{i_n}}(\rho, M) = \lim_{i_n=\infty} \frac{1}{4\pi\rho^2} \int_{C(\rho,M)} U_{i_n}(P)dP,$$

the last quantity being $A_{U_0}(\rho, M)$ since the $U_{i_n}(P)$ form a monotone-decreasing sequence with limit $U_0(P)$, for all P. Hence

$$A_U(\rho, M) = A_{U_0}(\rho, M).$$

From this equation and (17), §4, follows a similar result for the operation $a_U(\rho, M)$.

We may speak of the process just described in terms of a monotone-decreasing sequence as a *general sweeping-out process*, and summarize the results in the following theorem.

THEOREM. *For the general sweeping-out process, in which* $U_1(M) \geqq U_2(M) \geqq \cdots \geqq U_0(M) = \lim\ (i = \infty)\ U_i(M)$, *and* $U(M)$ *is the potential of a distribution* $f(e)$ *defined by the weak convergence of a subsequence of the* $f_i(e)$ *on* F, *we have*

(2) $\qquad U(M) = U_0(M), \qquad$ *M not on F*,

(3) $\qquad U(M) \leqq U_0(M), \qquad$ *M on F*,

(4) $\qquad A_U(\rho, M) = A_{U_0}(\rho, M), \qquad a_U(\rho, M) = a_{U_0}(\rho, M), \qquad$ *M in W*.

The potential $U(M)$ *and the distribution* $f(e)$, *for sets e measurable Borel, are uniquely determined, independently of the subsequence on which there is weak convergence.*

In fact, by (9), §2.2, and this equation (4), letting ρ approach zero, it follows from the uniqueness of $U_0(M)$ that there is only one possible function

$U(M)$. But given the potential $U(M)$ its mass distribution $f(e)$ is uniquely determined on all sets measurable Borel.*

In particular, as a further consequence of (4), letting ρ approach zero, $U(M)=U_0(M)$ wherever the latter is the point set derivative of its spatial integral. Hence if E is any set of positive spatial Lebesgue measure, we shall have

$$\int_E U(M)dM = \int_E U_0(M)dM. \tag{4'}$$

COROLLARY. *If $U_1(M)=U_1'(M)+U_1''(M)$, $U_1'(M)$ and $U_1''(M)$ being potentials of distributions of positive masses on F, and the generalized sweeping-out process is carried out separately on $U_1'(M)$ and $U_1''(M)$, then $U_i(M)=U_i'(M)+U_i''(M)$ determines a sweeping-out process for $U_1(M)$, so that $U(M)=U'(M)+U''(M)$.*

12. **Poincaré sweeping-out process for continuous potentials.** As a first case, we consider that discussed in the main by de la Vallée Poussin, in which a given potential $U(M)$ of a distribution of positive mass on F is assumed to be continuous on the set $\Sigma+s$ of §1, with respect to $\Sigma+s$ itself. We define a decreasing sequence $V_n(M)$ in terms of the sequence solution for the domain Σ and the boundary values $U(P)$, P on s, and describe the process of removing the mass from Σ as the *Poincaré sweeping-out process*.

More precisely, let Σ_n be a sequence of nested regular domains approximating to Σ,† and choose $V_n(M)$ as the following uniquely defined function:

$V_n(M)$ is continuous in W,

$V_n(M)$ is the solution of Laplace's equation in Σ_n which takes on the given values $U(P)$ on s_n, regular at ∞ if Σ is unbounded,

$V_n(M)=U(M)$ for M in $C\Sigma_n$, the complement of Σ_n.

Then for M in Σ, $V_0(M)=\lim\ (n=\infty)\ V_n(M)$ is the desired sequence solution, and is independent of the choice of the set of nested domains.‡

Since $V_n(M)$ is $\leqq U(M)$ and is harmonic wherever $V_n(M)<U(M)$, it is, being continuous, superharmonic, and, by §2, a potential of a positive mass

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* F. Riesz, Memoir (2), cited in §2. See also G. C. Evans, *Fundamental points of potential theory*, Rice Institute Pamphlet, vol. 7 (1920), pp. 252–329, p. 271 and p. 285, where the determination of the additive function of point sets is given in terms of a uniquely determined function of curves, with regular discontinuities.

† That is, Σ contains Σ_n with its boundary, and Σ_n contains Σ_{n-1} with its boundary; and the boundary of Σ_n is regular. (Hence there is one and only one solution of the Dirichlet problem for Σ_n which takes on continuously assigned boundary values which are continuous.) Every point of Σ is to lie ultimately in some Σ_n.

‡ O. D. Kellogg, *Foundations of Potential Theory*, Berlin, 1929, p. 317 ff.

distribution $f_n(e)$. Moreover $V_{n+1}(M)$ is everywhere $\leqq V_n(M)$, since the two functions are identical except in Σ_{n+1}, while in that region $V_n(M)$ is superharmonic and $V_{n+1}(M)$ harmonic. The sequence is therefore a special case of the sequence of §11, and the mass functions converge in the weak sense for a subsequence $\{n_i\}$ to a mass distribution $\mu(e)$ whose potential $V(M)$ is dominated by $V_0(M)$.

If Σ is a bounded domain, the total mass $f_n(W)$ or $f_n(C\Sigma_n)$ is $f(W)=f(F)$, for we have $V_n(M)=U(M)$ outside of a properly chosen sphere; if Σ is an infinite domain the $f_n(W)$ is $\leqq f(F)$, and some of the mass may be described as lost to infinity. In the limiting distribution there is no mass in Σ, and $\mu(s+B)=f(F)$ or is $\leqq f(F)$ according as Σ is bounded or unbounded; in the two cases the Poincaré sweeping-out process may be described as a transfer of the mass from Σ to its boundary s, or in part to s and in part to infinity.* In fact, from what is given above, it follows evidently that if $\nu_n(e)$ denotes the distribution $f_n(e\cdot s_n)$ and $\nu(e)$ a limiting distribution in weak convergence for the subsequence $\{n_i\}$, then

$$\begin{aligned} \nu(s) &= f(\Sigma), && \text{if } \Sigma \text{ is bounded,} \\ &\leqq f(\Sigma), && \text{if } \Sigma \text{ is unbounded.} \end{aligned}$$

Under this first case we may include that where $U(M)$ is continuous in the part of $\Sigma+s$ within a distance δ of s, for after a certain n the boundaries s_n will all lie within that neighborhood of s.

The following statement is immediate, as a property of the sequence solution. *Given two potentials $U'(M)$, $U''(M)$ of the kind just specified such that $U'(M)\geqq U''(M)$ for all M; then the corresponding functions $V_0'(M)$, $V_0''(M)$ satisfy the relation*

$$V_0'(M) \geqq V_0''(M), \qquad \textit{for all } M. \tag{5}$$

12.1. **Sweeping out of discontinuous potentials.** Turn now to the general case, $U(M)$ being the potential of an arbitrary distribution of positive mass on F, and take a sequence of nested domains Σ_n as in §12. Since $U(M)$ is lower semicontinuous and positive there exists a sequence $U^{(p)}(M)$ of not negative functions, defined and continuous on s_n and tending to $U(M)$ at every point of s_n. Let $U_n^{(p)}(M)$ be the function which is defined and continuous in Σ_n+s_n, harmonic in Σ_n, regular at ∞ if Σ is unbounded, and takes on the values $U^{(p)}(M)$ continuously on s_n. Define

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* In de la Vallée Poussin, loc. cit., the process is described in terms of an actual transfer of mass for a domain of sufficiently smooth boundary, based on its approximation by a domain consisting of a finite number of spheres.

$$(6)\qquad \begin{aligned} V_n(M) &= \lim_{p=\infty} U_n^{(p)}(M), &\quad M \text{ in } \Sigma_n,\\ &= U(M), &\quad M \text{ in } C\Sigma_n. \end{aligned}$$

The function $V_n(M)$, in Σ_n, is independent of the choice of the monotone-increasing sequence $U^{(p)}(M)$; this is in fact well known. We note that if we define $U^{(p)}(M)$, for all M, as the average $U(\rho, M)$, $\rho = 1/p$, it follows by §2 that $U^{(p)}(M)$ is a (continuous) potential of a distribution of positive mass on a set bounded independently of p, the total mass being $f(F)$. The function $V_n^{(p)}(M)$, equal to $U_n^{(p)}(M)$ for M in Σ_n and equal to $U^{(p)}(M)$ for M in $C\Sigma_n$, is therefore a potential of positive mass on a set which is bounded independently of n, p. But the functions $V_n^{(p)}(M)$ form a monotone-increasing sequence with respect to p, and $V_n(M) = \lim\ (p=\infty)\ V_n^{(p)}(M)$ for all M. Hence by §2.1, $V_n(M)$ is a potential of a distribution of positive mass on a bounded portion of $C\Sigma_n$, and the total mass is in value $f(F)$ or $\leqq f(F)$, according as Σ is bounded or unbounded. The same remark applies to a mass distribution to which these converge weakly as n tends to infinity.

The functions U, V_1, V_2, $\cdots$ constitute the decreasing sequence of §11. In fact, $V_{n+1}^{(p)}(M) \leqq V_n^{(p)}(M) \leqq U^{(p)}(M)$ for every p.

THEOREM. *The functions $V_0(M)$, $V(M)$ and the mass distribution $\mu(e)$, for sets e measurable Borel, are uniquely determined, independently of the choice of the sequence of nested regular domains for Σ, and of the subsequence over which the weak convergence is established.*

Consider first two regular domains Σ_1, Σ_2 such that Σ contains Σ_2, which contains Σ_1. We note that the corresponding potentials V_1, V_2 satisfy the relation

$$V_1(M) \geqq V_2(M), \qquad M \text{ in } W.$$

In fact, given M, and choosing $U^{(p)}(M) = U(\rho, M)$, $\rho = 1/p$, as above,

$$V_1^{(p)}(M) \geqq V_2^{(p)}(M), \qquad \text{for all } p.$$

Let now $\{\Sigma_n'\}$, $\{\Sigma_m''\}$ constitute two sequences of nested regular domains for Σ. We may construct a third sequence of such domains, $\{\Sigma_j^0\}$, which contains an infinite number of domains of each of the sequences $\{\Sigma_n'\}$, $\{\Sigma_m''\}$; for, given any Σ_n', there is a Σ_m'' which contains Σ_n' and its boundary. In fact, for every m the set $(\Sigma_n' + s_n')\cdot(C\Sigma_m'')$ is closed and contains its successor when m is replaced by $m+1$; but since there is no point belonging to it for all m it must become empty for m sufficiently large.

It follows that the limiting functions $V_0'(M)$, $V_0''(M)$ are identical. In fact,

$$V_0'(M) = \lim_{n=\infty} V_n'(M) = \lim_{j=\infty} V_j^0(M) = \lim_{m=\infty} V_m''(M) = V_0''(M).$$

Hence the function $V_0(M)$ is unique.

But also, if $V(M)$ is the potential resulting from the weak convergence on any subsequence $\{\Sigma_{n_i}\}$ of any sequence $\{\Sigma_n\}$, we have, by (9), §2.2, and (4), §11,

$$V(M) = \lim_{\rho=0} A_V(\rho, M) = \lim_{\rho=0} A_{V_0}(\rho, M), \tag{6'}$$

where the function $V_0(M)$ is unique. Hence $V(M)$ is unique. Finally, if $V(M)$ is unique, the mass distribution $\mu(e)$, of which it is the potential, is uniquely determined on all sets measurable Borel.

We note finally that the inequality (5) is still valid. That is, if the potentials $U'(M)$, $U''(M)$, of positive mass distributions, are given, with $U'(M) \geqq U''(M)$, for all M, then $V_0'(M) \geqq V_0''(M)$ for all M. Moreover, for the resulting potentials $V'(M)$, $V''(M)$ we have, by means of (6'),

$$V'(M) \geqq V''(M), \qquad M \text{ in } W. \tag{5'}$$

12.2. **Alternative procedure for sweeping-out of discontinuous potentials.** The following method also extends the Poincaré sweeping-out process to apply to an arbitrary potential, and is more in line with the procedure of de la Vallée Poussin for continuous potentials. We write the given potential $U(M)$ in the form

$$U(M) = U'(M) + U''(M),$$

$$U'(M) = \int_W \frac{1}{MP}\, df(e_P\cdot\Sigma), \qquad U''(M) = \int_W \frac{1}{MP}\, df(e_P\cdot(s + B)),$$

and carry out the process on $U'(M)$. For this purpose we form a monotone-increasing sequence of continuous potentials $U'^{(p)}(M)$ of positive distributions on a bounded set (or potentials each continuous in a portion of $\Sigma+s$ neighboring s), such that

$$\lim_{p=\infty} U'^{(p)}(M) = U'(M), \qquad M \text{ in } W.$$

Let $V_0'^{(p)}(M)$, $V'^{(p)}(M)$ be the function and potential, respectively, generated by the sweeping-out of the mass from Σ of the continuous potentials $U'^{(p)}(M)$.

The functions $V_0'^{(p)}(M)$ form a monotone-increasing sequence dominated by $U'(M)$. Hence there exists the function

$$\overline{V}_0(M) = U''(M) + \overline{V}_0'(M) \tag{7}$$

with

$$\overline{V}_0'(M) = \lim_{p=\infty} V_0'^{(p)}(M). \tag{7'}$$

Moreover the total mass for $U'^{(p)}(M)$ is bounded, $\leqq f(\Sigma)$, and lies on a set which is bounded independently of p, and any closed set contained in B, ultimately, for sufficiently large p, bears no mass. Consequently the mass functions for $V'^{(p)}(M)$ converge in the weak sense, as p tends to ∞ on a subsequence, to a positive mass distribution $\bar{\mu}(e)$ which lies entirely on s. We define $\overline{V}(M)$ as the potential

$$\overline{V}(M) = U''(M) + \overline{V}'(M) \tag{8}$$

with

$$\overline{V}'(M) = \int_W \frac{1}{MP}\, d\bar{\mu}(e_P). \tag{8'}$$

Lemma I. *The function $\overline{V}_0(M)$ is independent of the choice of the sequence $U'^{(p)}(M)$.*

This lemma is verified by means of the relation (5) when the monotone-increasing sequence $U'^{(p)}(M)$ has been replaced by the strictly increasing sequence of potentials $(1-1/p)\ U'^{(p)}(M)$. Two such sequences may then be compared in the customary manner. In fact, let $U'^{(p)}(M)$ be a strictly increasing sequence of such continuous potentials, $u'^{(p)}(M)$ a strictly increasing sequence of such potentials, each continuous in a closed region σ_p comprising s and the points of Σ not distant from s by more than some $\delta_p>0$. For p_1 given, since $u'^{(p)}(M)$ is lower semicontinuous the set of points where $u'^{(p)}(M) \leqq U'^{(p_1)}(M)$ is closed, and hence, for p sufficiently great, will vanish. Similarly for p_2 given, the set of points in σ_{p_2} where $U'^{(p)}(M) \leqq u'^{(p_2)}(M)$ is closed, and hence, for p sufficiently great, will vanish. Accordingly, for the corresponding $V'^{(p)}(M)$, $v_0'^{(p)}(M)$, obtained by the sweeping out, we have the analogous relations, by (5), and both sequences $V_0'^{(p)}(M)$, $v_0'^{(p)}(M)$ have the same limiting function $\overline{V}_0(M)$.

Lemma II. *The equation* (4), §11, *remains valid for the procedure of the present* §12.2.

In fact, we have merely to repeat the proof already given of (4).

Theorem. *Given $U(M)$, the potential of an arbitrary distribution of positive mass on F, the limiting functions $V_0(M)$, $\overline{V}_0(M)$, determined by the processes of* §§12.1, 12.2 *respectively, are identical:*

$$V_0(M) = \overline{V}_0(M), \qquad \textit{for all } M \textit{ in } W. \tag{9}$$

The statement is true for M in $s+B$. For, for M in $s+B$,

$$V_0(M) = U(M),$$
$$\overline{V}_0'(M) = \lim_{p=\infty} V_0'^{(p)}(M) = \lim_{p=\infty} U'^{(p)}(M) = U'(M),$$
$$\overline{V}_0(M) = \overline{V}_0'(M) + U''(M) = U'(M) + U''(M) = U(M).$$

For M in Σ, we have

$$U''(M) + V_0'^{(p)}(M) \leqq U''(M) + V_n'^{(p)}(M) \leqq V_n(M), \qquad n, p \text{ arbitrary},$$
$$\overline{V}_0(M) = U''(M) + \lim_{p=\infty} V_0'^{(p)}(M) \leqq V_n(M), \qquad n \quad \text{arbitrary}.$$

Hence

$$\overline{V}_0(M) \leqq V_0(M).$$

In order to establish the complementary inequality, let σ_δ be the portion of Σ at a distance from s not greater than δ, and Σ_δ the remaining open set. Write

$$U'(M) = U_\delta(M) + U'''(M)$$

where

$$U_\delta(M) = \int_W \frac{1}{MP}\, df(e_P\cdot\Sigma_\delta), \qquad U'''(M) = \int_W \frac{1}{MP}\, df(e_P\cdot\sigma_\delta).$$

Given $\epsilon>0$, and Q a fixed point in Σ, distant an amount κ from s, we choose a positive $\delta<\kappa$ so that $f(\sigma_\delta)/\kappa$ shall be $<\epsilon$; this is possible, since, Σ being an open set, $\lim\ (\delta=0)\ f(\sigma_\delta)=0$. But, evidently, with notation corresponding to that just used,

$$V_0(Q) = U''(Q) + V_{\delta 0}(Q) + V_0'''(Q) < U''(Q) + V_{\delta 0}(Q) + \epsilon,$$
$$\overline{V}_0(Q) = U''(Q) + \overline{V}_{\delta 0}(Q) + \overline{V}_0'''(Q) \geqq U''(Q) + \overline{V}_{\delta 0}(Q).$$

Now if we denote by $U_\delta^{(p)}(M)$ the average of $U_\delta(M)$ over a sphere of radius $\rho=1/p$, and take $p>1/\delta$, we shall have $U_\delta^{(p)}(M)=U_\delta(M)$ for M near enough to s. Hence, for the corresponding harmonic functions determined by their continuous values on s_n,

$$V_{\delta n}^{(p)}(Q) = V_{\delta n}(Q), \qquad n \text{ great enough},$$

whence

$$V_{\delta 0}^{(p)}(Q) = V_{\delta 0}(Q), \qquad \overline{V}_{\delta 0}(Q) = \lim_{p=\infty} V_{\delta 0}^{(p)}(Q) = V_{\delta 0}(Q).$$

Consequently, substituting in the above inequalities,

$$\overline{V}_0(Q) \geqq U''(Q) + V_{\delta 0}(Q) > V_0(Q) - \epsilon.$$

This however yields the desired complementary inequality $\overline{V}_0(Q) \geqq V_0(Q)$, whence $\overline{V}_0(Q) = V_0(Q)$ for Q in Σ.

COROLLARY. *The potential $\overline{V}(M)$ is uniquely determined, and is the same as the potential $V(M)$ of* §12.1,

$$\overline{V}(M) = V(M), \qquad M \text{ in } W. \tag{10}$$

In fact, by (4), §11,

$$A_{\overline{V}}(\rho, M) = A_V(\rho, M) = A_{V_0}(\rho, M),$$

whence the conclusion follows by letting ρ approach zero.

In particular, the process of §12.2 is instanced in the sweeping out of a general positive distribution on Σ by sweeping out successively the portions within the domains Σ_k, where these constitute a sequence of nested domains for Σ.

13. **Consistency theorems.** We may compare the potentials resulting from the succession of a generalized and a Poincaré sweeping-out process, or of two Poincaré sweeping-out processes.

LEMMA. *Let $U_1, U_2, \cdots$ be a monotone-decreasing sequence of potentials of positive mass distributions on F, with limit $U_0(M)$; let $f(e)$ be the mass distribution to which a subsequence of the mass distributions $f_i(e)$ of $U_i(M)$ converges in the weak sense, and $U(M)$ its potential. Let $V_{1,0}(M)$, $V_{2,0}(M), \cdots, V_0(M)$ be the limiting functions obtained by the sweeping out of the mass distributions $f_1(e), f_2(e), \cdots, f(e)$ from Σ; then*

$$V_0(M) = \lim_{i=\infty} V_{i,0}(M), \qquad M \text{ in } \Sigma. \tag{11}$$

By (5), the functions $V_{i,0}(M)$ form a monotone-decreasing sequence, with limit, say, $\overline{V}_0(M)$. We show that $\overline{V}_0(M) = V_0(M)$, M in Σ. With the aid of (2) and (3), §11, we have

$$U(M) \leqq U_0(M) \leqq U_i(M), \qquad i = 1, 2, \cdots, M \text{ in } \Sigma.$$

Hence $V_0(M) \leqq V_{i,0}(M)$ and

$$V_0(M) \leqq \overline{V}_0(M), \qquad M \text{ in } \Sigma. \tag{12}$$

In order to establish the complementary inequality, let Σ_n be a set of nested regular domains for Σ, with the boundaries s_n of the Σ_n taken smooth enough so that $\lambda_n(M, P)$, the normal derivative of the Green's function for the Σ_n, with pole at M, will be continuous in P, for P on s_n. We carry out the

Poincaré process of §12.1 in terms of this set of nested domains, taking the monotone-increasing sequences $U_i^{(p)}(M)$ of that section as continuous potentials of positive mass. But for M in Σ_n,

$$V_{i,n}^{(p)}(M) = \frac{1}{4\pi}\int_{s_n} \lambda_n(M, P)U_i^{(p)}(P)dP,$$

whence

$$V_{i,n}(M) = \frac{1}{4\pi}\int_{s_n} \lambda_n(M, P)U_i(P)dP. \tag{13}$$

Similarly for the same process carried out on $U(M)$,

$$V_n(M) = \frac{1}{4\pi}\int_{s_n} \lambda_n(M, P)U(P)dP.$$

We note that we have also

$$V_n(M) = \frac{1}{4\pi}\int_{s_n} \lambda_n(M, P)U_0(P)dP. \tag{13'}$$

In fact, if σ is any regular surface element,

$$\int_\sigma U_0(P)dP = \lim_{i=\infty}\int_\sigma U_i(P)dP,$$

$$\int_\sigma U(P)dP = \int_W df(e_R)\int_\sigma \frac{1}{PR}\,dP.$$

The inside integral of the right hand member is however the potential at R of unit density distribution on σ, and is therefore continuous in R, for R in W. Hence by the weak convergence property there is the subsequence $\{i'\}$ of $\{i\}$ such that

$$\int_W df(e_R)\int_\sigma \frac{1}{PR}\,dP = \lim_{i'=\infty}\int_W df_{i'}(e_R)\int_\sigma \frac{1}{PR}\,dP.$$

Accordingly,

$$\int_\sigma U(P)dP = \lim_{i'=\infty}\int_\sigma U_{i'}(P)dP$$

and $\int_\sigma U(P)dP = \int_\sigma U_0(P)dP$; from which (13′) follows.

We have now what we need. Given Q in Σ and $\epsilon>0$, we can find a stage n_1, of the Poincaré process, such that for $n \geqq n_1$, we have

$$V_0(Q) > V_n(Q) - \epsilon.$$

Consequently, since by (13), (13′), $V_n(Q) = \lim\ (i=\infty)\, V_{i,n}(Q)$,

$$V_0(Q) > \lim_{i=\infty} V_{i,n}(Q) - \epsilon,$$

and thus

$$V_0(Q) > \lim_{i=\infty} V_{i,0}(Q) - \epsilon,$$

for $V_{i,0}(Q) \leqq V_{i,n}(Q)$, the $V_{i,n}(M)$ forming a monotone-decreasing sequence in n, according to the definition of the Poincaré process as applied to $U_i(M)$. But then, $V_0(Q) > \overline{V}_0(Q) - \epsilon$ and

$$V_0(Q) \geqq \overline{V}_0(Q), \qquad Q \text{ in } \Sigma.$$

This is the complementary inequality, and the lemma is therefore established.

Theorem I. *Let $U_1, U_2, \cdots$ be a monotone-decreasing sequence of potentials of positive mass distributions on F, generating a potential $U(M)$ by the generalized sweeping-out process, and let $v_1, v_2, \cdots, V$ be the potentials arising from the sweeping out of the above masses from Σ. Then $v_1, v_2, \cdots$ also constitute a generalized sweeping-out process of monotone-decreasing potentials, generating the same potential $V(M)$, for all M.*

By (5′), §12.1, the potentials $v_i(M)$ constitute a monotone-decreasing sequence. Let then E be any bounded set of positive spatial Lebesgue measure. With reference to the notation of the lemma, and writing $G = C\Sigma$, as before, we have by (4′), §11,

$$\begin{aligned}\int_E v_i(M)dM &= \int_E V_{i0}(M)dM \\ &= \int_{E\cdot G} V_{i0}(M)dM + \int_{E\cdot\Sigma} V_{i0}(M)dM.\end{aligned}$$

We denote $\lim\ (i=\infty)\ v_i(M)$ by $v_0(M)$ and the corresponding potential by $v(M)$, and by means of (4′), we obtain

$$\begin{aligned}\int_E v(M)dM &= \int_E v_0(M)dM = \lim_{i=\infty} \int_E v_i(M)dM \\ &= \lim_{i=\infty} \int_{E\cdot G} V_{i0}(M)dM + \lim_{i=\infty} \int_{E\cdot\Sigma} V_{i0}(M)dM.\end{aligned}$$

But in $E\cdot G$, $V_{i0}(M) = U_i(M)$, and in $E\cdot\Sigma$, $\lim\ (i=\infty)\ V_{i0}(M) = V_0(M)$ by the lemma of this section. Hence, since we are dealing with monotone sequences,

$$\int_E v(M)dM = \int_{E\cdot G} U_0(M)dM + \int_{E\cdot\Sigma} V_0(M)dM.$$

But now again we may apply (4′), and write

$$\int_{E\cdot G} U_0(M)dM = \int_{E\cdot G} U(M)dM = \int_{E\cdot G} V_0(M)dM,$$

so that

$$\int_E v(M)dM = \int_E V_0(M)dM = \int_E V(M)dM.$$

In particular,

$$a_v(\rho, M) = a_V(\rho, M),$$

whence, letting ρ approach zero,

$$v(M) = V(M), \qquad M \text{ in } W.$$

This is what was to be proved.

In particular, we may take, for the potentials $U_i(M)$, the sequence of potentials obtained by sweeping out a given potential $U_1(M)$ from Σ by means of a sequence of nested regular domains Σ_i for Σ, and for the $v_i(M)$, the sequence of potentials arising from the sweeping out of the $U_i(M)$ from a domain Σ' of which the boundary s' is a closed subset of $G=C\Sigma$. By the theorem of §12.1 the functions $v_1(M)$, $v_2(M)$, $\cdots$ are all identical. We deduce then, from the theorem of the present section, that the potential arising by sweeping out $U_1(M)$ from Σ' is the same as that obtained by first sweeping out $U_1(M)$ from Σ and then sweeping out the resulting potential from Σ'.

THEOREM II. *Let g be a closed subset of $G=C\Sigma$, and s' the external frontier of g, so that s' is the boundary of an infinite domain Σ' which contains Σ. Let $U(M)$ be a potential of a positive mass distribution $f(e)$ on F. Then the potential arising from the sweeping out of $f(e)$ from Σ' is everywhere the same as that obtained first, by sweeping out $f(e)$ from Σ, and second, by sweeping out the resulting distribution from Σ'.*

14. **Sweeping out of unit mass.** Consider a distribution $\mu(e, Q)$ arising from a sweeping out of unit mass at Q from the domain Σ of §1, and denote by $v_0(M, Q)$, $v(M, Q)$ the corresponding limiting and potential functions. For definiteness we take Σ as bounded; the unbounded region may be treated in the same manner.

As a first case we assume that Σ is normal for the Dirichlet problem (that is, corresponding to arbitrary continuous values assigned on s we assume that there exists in Σ a solution of Laplace's equation which takes on continuously the assigned boundary value at every point of s), and that the boundary s is sufficiently smooth for applications of Green's theorem; in fact, that the normal derivative of the Green's function for Σ, with pole at Q in Σ, is continuous on s. We denote this derivative by $\lambda(Q, P)$. It is harmonic in Q for Q in Σ.

The function*

$$I = \frac{1}{4\pi}\int_s \frac{\lambda(Q, P)}{MP}\,dP \tag{14}$$

is harmonic as a function of M for M not on s, and is continuous in M for all M, vanishing continuously at ∞. For M fixed in $C(\Sigma+s)=B$, I is the value at Q of the function, harmonic in Σ, which takes on the value $1/(MP)$ as Q tends to a point P on s. Hence for Q in Σ, M in B,

$$I = 1/(QM),$$

and since both members are continuous, the same equation holds for M on s. Consequently, in Σ, as a function of M, I is the harmonic function which takes on continuously the values $1/(QP)$ as M tends to P on s. We deduce then that

$$v(M, Q) = v_0(M, Q) = \frac{1}{4\pi}\int_s \frac{\lambda(Q, P)}{MP}\,dP, \qquad M \text{ in } W. \tag{15}$$

The distribution of mass is uniquely determined on every set measurable Borel, if its potential is everywhere given. Hence, for our surface s, we have†

$$\mu(e, Q) = \frac{1}{4\pi}\int_{e\cdot s} \lambda(Q, P)dP. \tag{15'}$$

This is an absolutely continuous distribution of mass on s whose surface density at a point P of s is the normal derivative of the Green's function with pole at Q, divided by 4π. From (15), the Green's function itself is given by the equation

$$g(Q, M) = \frac{1}{QM} - v(M, Q). \tag{15''}$$

Let now $f(e)$ denote an arbitrary distribution of positive mass, lying on a

* A generalization of the corresponding function for the circle; see Picard, *Traité d'Analyse*, vol. 2, Paris, 1905, p. 91.

† The corresponding relation in two dimensions interprets the fact that the method of conformal mapping applies to the sweeping out of unit mass in the same way as to the normal derivative of the Green's function. See C. de la Vallée Poussin, *Extension de la méthode du balayage de Poincaré, et problème de Dirichlet*, Annales de l'Institut Henri Poincaré, vol. 2 (1932), pp. 169–232, at p. 190.

closed set in Σ, with a potential $U(M)$ which is therefore continuous on $C(\Sigma)=s+B$. The function $V_0(M)$ which is harmonic in Σ, identical with $U(M)$ in $C(\Sigma+s)=B$, and takes on continuously the values $U(M)$ on s, for approach from Σ, is therefore, by the mean value property (3), §11, identical with the potential $V(M)$ of the swept-out mass $\mu(e)$. This mass lies entirely on s. We have

$$\begin{aligned} V(M) = V_0(M) &= \frac{1}{4\pi}\int_W df(e_Q)\int_s \lambda(Q, P)\frac{1}{MP}dP \\ &= \int_W v(M, Q)df(e_Q). \end{aligned} \tag{16}$$

In fact, this last integral is a continuous function of M, $v(M, Q)$ being a continuous function of M in W and of Q in the closed set on which $f(e)$ lies. Moreover, for M in $s+B$, $v(M, Q)$ is $1/(MQ)$, so that the given integral reduces to $U(M)$. It is also harmonic in M for M in Σ, since $v(M, Q)$ has that property.

The function $\lambda(Q, P)$ is not negative, and therefore we may change the order of integration in (16) and write

$$V(M) = \frac{1}{4\pi}\int_s \frac{dP}{MP}\int_W \lambda(Q, P)df(e_Q). \tag{17}$$

That is, $V(M)$ is the potential of the distribution of positive mass

$$\begin{aligned} \mu(E) &= \frac{1}{4\pi}\int_{E\cdot s} dP\int_W \lambda(Q, P)df(e_Q) \\ &= \frac{1}{4\pi}\int_W df(e_Q)\int_{E\cdot s}\lambda(Q, P)dP = \frac{1}{4\pi}\int_\Sigma df(e_Q)\int_{E\cdot s}\lambda(Q, P)dP. \end{aligned} \tag{18}$$

The mass distribution is absolutely continuous on s, of surface density

$$\frac{1}{4\pi}\int_\Sigma \lambda(Q, P)df(e_Q),$$

and from (15′), (18)

$$\mu(E) = \int_\Sigma \mu(E, Q)df(e_Q). \tag{19}$$

This last equation includes as a special case the following, where s_1 is a regular surface bounding a domain Σ_1 interior to Σ, $\mu_\Sigma(e, Q)$ and $\mu_{\Sigma_1}(e, Q)$ denoting the respective swept-out unit masses:*

* Equation (19′) is given in the case of smooth boundaries by de la Vallée Poussin, loc. cit., p. 182.

$$\mu_\Sigma(E, P) = \int_\Sigma \mu_\Sigma(E, Q) d\mu_{\Sigma_1}(e_Q, P). \tag{19'}$$

We are able to extend the equation (19), and therefore of course (19′), to a general domain Σ, whose boundary is a closed bounded set. For the sake of definiteness we retain the hypothesis that Σ is a bounded set.

THEOREM. *Let $f(e)$ be a distribution of positive mass on a general* (*bounded*) *domain Σ whose boundary is s. If $\mu(e)$, $\mu(e, Q)$ are the mass distributions obtained by the sweeping out of $f(e)$ and of unit mass at Q, respectively, then* (19) *is valid.*

Suppose first that $f(e)$ is a distribution lying entirely on a closed set F interior to Σ; without loss of generality we may suppose F to be perfect. Let Σ_n be a sequence of nested regular domains for Σ, and $\mu_n(e)$, $\mu_n(e, Q)$ the sweeping-out distributions satisfying (19); let

$$\bar{\mu}(e) = \int_W \mu(e, Q) df(e_Q).$$

LEMMA I. *If the mass distribution $f(e)$ is swept out of Σ, by means of the domains Σ_n, then, for $\phi(P)$ continuous,*

$$\lim_{n=\infty} \int_W \phi(P) d\mu_n(e_P) \text{ exists, and equals } \int_W \phi(P) d\mu(e_P). \tag{20}$$

Otherwise there would be a subsequence $\{n_i\}$ such that $\int_W \phi d\mu_{n_i}$ would approach some value different from the right hand member. But this is impossible, since there would be a subsequence of the $\{n_i\}$ for which the mass distributions would converge weakly to a swept-out distribution, and the swept-out distribution is unique.

LEMMA II. *For each set E, measurable Borel, the function $\mu(E, Q)$ is harmonic in Q, for Q in Σ, and is $\leqq 1$.*

In fact, for Σ_n as above and $\phi(P)$ continuous, from Lemma I,

$$\lim_{n=\infty} \int_W \phi(P) d\mu_n(e_P, Q) = \int_W \phi(P) d\mu(e_P, Q).$$

The right hand member is harmonic in Q, for Q in Σ, for each continuous $\phi(P)$; for the integral of the left hand member is harmonic in Q, Q in Σ_n, and converges to the right hand member, remaining bounded.

Consequently if $\psi(P)$ is any bounded function, measurable Borel, the I-

integral $\int_W \psi(P)d\mu(e_P, Q)$ is harmonic in Q, Q in Σ. In fact, such a function is a (transfinite) limit, starting from continuous functions $\phi(P)$. In particular, if we take $\psi(P)=1$ on E and 0 elsewhere, the I-integral reduces to $\mu(E, Q)$, and this quantity is therefore harmonic in Q, Q in Σ.

Finally, $\mu(E, Q) \leqq 1$, since $\mu_n(W, Q) \leqq 1$.

LEMMA III.* *A sufficient condition that $\mu_n(e)$ converges to $\bar{\mu}(e)$ weakly is that*

$$\lim_{n=\infty} \int_W \phi(P)d\mu_n(e_P) = \int_W \phi(P)d\bar{\mu}(e_P)$$

for every continuous $\phi(P)$.

Consider in fact a rectangular net the boundaries of whose meshes bear none of the mass distribution $\bar{\mu}(e)$. Let ω be one such open mesh, Ω its closed cover. Let $\phi_1(P)=1$ in Ω and 0 at a distance $\geqq 1/k$ from ω, being continuous, $\leqq 1$, in W. Given $\epsilon>0$, by taking k large enough, we have

$$\int_W \phi_1(P)d\bar{\mu}(e) < \bar{\mu}(\omega) + \epsilon,$$

$$\limsup_{n=\infty} \mu_n(\Omega) \leqq \lim_{n=\infty} \int_W \phi_1(P)d\mu_n(e_P) < \bar{\mu}(\omega) + \epsilon.$$

On the other hand, if we take $\phi_2(P)$ continuous and $\leqq 1$ in W, zero outside ω, and unity in ω at a distance $\geqq 1/k$ from $C\omega$, we have similarly, taking k large enough,

$$\int_W \phi_2(P)d\bar{\mu}(e_P) > \bar{\mu}(\omega) - \epsilon,$$

$$\liminf_{n=\infty} \mu_n(\omega) \geqq \lim_{n=\infty} \int_W \phi_2(P)d\mu_n(e_P) > \bar{\mu}(\omega) - \epsilon.$$

In other words, $\mu_n(e)$ converges on each mesh of the net to $\bar{\mu}(e)$.

To return to the theorem, we take $\phi(P)$ continuous in W, and obtain

$$\begin{aligned}\int_W \phi(P)d\mu_n(e_P) &= \int_W \phi(P)d_P\left[\int_W \mu_n(e_P, Q)df(e_Q)\right]\\ &= \int_W \phi(P)d_P\left[\int_F \mu_n(e_P, Q)df(e_Q)\right].\end{aligned}$$

But $\mu_n(W, Q) \leqq 1$, $\mu_n(e, Q)$ is continuous in Q for Q in F, and $\phi(P)$ is continuous; hence we may change the order of integration and write†

* See §2.1, footnote to (7).

† G. C. Evans, *Functionals and their Applications*, New York, 1918, p. 103.

$$\int_W \phi(P) d\mu_n(e_P) = \int_F df(e_Q) \int_W \phi(P) d\mu_n(e_P, Q).$$

Similarly,

$$\int_W \phi(P) d\bar{\mu}(e_P) = \int_F df(e_Q) \int_W \phi(P) d\mu(e_P, Q).$$

But this again, from the weak convergence on $\{n\}$, is equal to

$$\int_F df(e_Q) \lim_{n=\infty} \int_W \phi(P) d\mu_n(e_P, Q).$$

The function $\int_W \phi(P) d\mu_n(e_P, Q)$ is bounded, irrespective of n, is harmonic in Q for Q in F, and in fact approaches its limit uniformly for Q in F. Hence

$$\int_W \phi(P) d\bar{\mu}(e_P) = \lim_{n=\infty} \int_F df(e_Q) \int_W \phi(P) d\mu_n(e_P, Q)$$
$$= \lim_{n=\infty} \int_W \phi(P) d\mu_n(e_P).$$

By Lemma III then, $\mu_n(e)$ converges weakly to $\bar{\mu}(e)$, and therefore $\bar{\mu}(e)$ and $\mu(e)$ are identical, and $\mu(e)$ is given by (19).

In order to complete the proof of the theorem, let now $f(e)$ be any positive mass distribution, finite in total amount, on Σ. We have

$$f(e) = f(e \cdot F_\delta) + f(e \cdot [\Sigma - F_\delta])$$

where F_δ is the portion of Σ distant from s by at least as much as δ. For sufficiently large n, the region Σ_n contains in its interior any given F_δ, and therefore if we denote by $\mu_\delta(e)$ the mass distribution obtained by the sweeping out of the distribution $f(e \cdot F_\delta)$, we shall have

$$\mu_\delta(e) = \int_W \mu(e, Q) df(e_Q \cdot F_\delta).$$

But, according to the process of §12.2, the swept-out distribution for $f(e)$ is given by the formula

$$\mu(e) = \lim_{\delta=0} \mu_\delta(e)$$
$$= \lim_{\delta=0} \int_W \mu(e, Q) df(e_Q \cdot F_\delta),$$

and since

$$\int_W \mu(e, Q)df(e_Q) - \int_W \mu(e, Q)df(e_Q\cdot F_\delta) = \int_W \mu(e, Q)d[f(e_Q) - f(e_Q\cdot F_\delta)]$$
$$\leqq f(\Sigma) - f(\Sigma\cdot F_\delta),$$

this limit is precisely

$$\int_W \mu(e, Q)df(e_Q),$$

which is the fact which was to be proved.

V. Capacity and Kellogg's Lemma

15. **Conductor potential and capacity.** Let s be a closed bounded set, the boundary of an *infinite* domain Σ. Let Σ_1 be an infinite domain contained in Σ, of which the boundary s_1 is bounded and regular, and let $\xi_1(M)$ be the function which is continuous in W, harmonic in Σ_1 (vanishing continuously at ∞) and equal to 1 on $C\Sigma_1$. Then $\xi_1(M)$ is evidently superharmonic, and, by §2, the potential of some distribution of positive mass. This mass lies entirely on s_1.

Let $\eta_0(M)$ be the limiting function obtained by the sweeping out of the mass of $\xi_1(M)$ from Σ; that is, in Σ, $\eta_0(M)$ is the sequence solution for the values 1 on s. Let $\eta(M)$ be the potential of a distribution of positive mass $\nu(e)$ arising from the sweeping out, and K the total mass of this distribution. Both $\eta_0(M)$ and $\eta(M)$ are independent of the choice of the sequence of nested regular domains Σ_n, and $\eta(M)=\eta_0(M)$ in Σ. Moreover, K depends merely on the values of $\eta(M)$ in Σ, and is therefore uniquely determined.

The distribution $\nu(e)$ may be called a *conductor distribution*, and its potential a *conductor potential*. The quantity K is called the *capacity* of the closed set s and of the closed set $G=s+B$, in fact, of any closed set g whose external frontier is s. This is the value of the capacity as defined by Wiener.* In order to complete the definition for sets E which are bounded and measurable Borel, but not necessarily closed, we may write

$$K(E) = K(\overline{E}),$$

where $\overline{E}$ is the closed cover of E. The capacity K may of course in special cases have the value zero.

But other definitions of capacity are possible. We define $K_a(E)$, $K_b(E)$, $K_c(E)$ as the upper bounds of total masses of positive mass distributions on E

* N. Wiener, *The Dirichlet problem*, Journal of Mathematics and Physics of the Massachusetts Institute of Technology, vol. 3 (1924), pp. 24–51; see §4. In this paper the author discusses weight, capacity and conductor potential and arrives at a determination of the conductor distribution, which he calls the "outer charge."

of which the potentials do not surpass unity on the following portions, respectively, of space:

(*a*) on the complement $C\overline{E}$ of the closed cover of E;

(*b*) on the complement of E;

(*c*) on the whole space.

We say that K_a, K_b, or K_c is zero if no distribution exists for which the corresponding upper bound of potential is finite. The quantity K_c is the capacity as defined by de la Vallée Poussin.*

Evidently $K_c \leqq K_b \leqq K_a$. But also, if s is the exterior frontier of E and Σ the infinite domain bounded by s, and if Σ_n form a sequence of regular nested domains for Σ, we shall have

$$K(E) = \lim_{n=\infty} K(C\Sigma_n).$$

But $K_a(E) \leqq K_a(\overline{E}) \leqq K(C\Sigma_n)$. Accordingly

$$K_a(E) \leqq \lim_{n=\infty} K(C\Sigma_n) = K(E).$$

Hence

$$K_c(E) \leqq K_b(E) \leqq K_a(E) \leqq K(E). \tag{1}$$

The following properties may be mentioned as familiar, or directly verifiable.

(2) If E is a single point, $K(E)=0$, and similarly for K_a, K_b, K_c.

(2′) If E_1 is contained in E_2, $K(E_1) \leqq K(E_2)$, and similarly for K_a, K_b, K_c.

(2″) If $E=E_1+E_2+\cdots$, and $K_c(E_i)=0$ for all i, then $K_c(E)=0$.

We have also the theorem of de la Vallée Poussin† that for closed bounded sets $K_c=K$.

THEOREM. *For closed bounded sets* $\overline{E}$,

$$K = K_a = K_b = K_c. \tag{3}$$

With regard to (1) it follows that we need merely prove that $K_c(\overline{E}) \geqq K(\overline{E})$. This fact is evident if $K(\overline{E})=0$. And if $K(\overline{E})>0$, any conductor distribution $\nu(e)$ is itself a distribution on $\overline{E}$ of which the potential $\eta(M)$ nowhere exceeds unity; that is to say, $K_c(\overline{E})$ is at least as great as $K(\overline{E})$.

For sets which are not closed, however, the various definitions of capacity are not all equivalent. For instance, if E_1 is a denumerable set of points dense everywhere within the sphere of radius 1/2, it follows from (2″) that $K_c(E_1)=0$. Similarly $K_b(E_1)=0$. But evidently $K(E_1)=1/2$; and also $K_a(E_1)=1/2$, since a point mass as near 1/2 in value as desired may be placed on a

* C. de la Vallée Poussin, loc. cit., p. 225.

† Ibid., p. 226.

point of E_1 so near the center of the sphere that the potential outside the sphere does not exceed unity. Also if E_2 is a denumerable set, everywhere dense on the surface of the sphere, we have $K_c(E_2)=0$, $K(E_2)=1$. But it is clear that $K_b(E_2)=0=K_a(E_2)$, since if there is a positive mass on E_2, there will be a positive mass on some point Q of E_2, and its potential will be greater than N, N given arbitrarily, in a neighborhood of Q. This neighborhood includes points not in the closed cover of E_2. Similar reasoning establishes the fact that if $E=E_1+E_2$, then

$$0 = K_c(E) = K_b(E) < K_a(E) = \tfrac{1}{2} < K(E) = 1.$$

15.1. **Capacity of sets measurable Borel.** We prove the following

THEOREM. *For any bounded set E measurable Borel, $K_b(E)=K_c(E)$.*

On account of (1) it is sufficient to show that $K_c(E) \geqq K_b(E)$, where $K_b(E)>0$. Suppose the contrary, that $K_b(E)>K_c(E)$. Then there exists a distribution of positive mass $\nu(e)$ on E such that $\nu(E)>K_c(E)$ and such that the potential $V_\nu(M)$ of this mass is $\leqq 1$ on CE. For $K_b(E)$ is the upper bound of such $\nu(E)$.

There exists a closed set F, contained in E, such that $\nu(F)$ differs as little as we please from $\nu(E)$; for E, being measurable Borel, belongs to a normal family for $\nu(e)$ in the sense of de la Vallée Poussin.* We may assume then that $\nu(F)>K_c(E)$. Let $\mu(e)=\nu(e\cdot F)$ and let $V_\mu(M)$ be the potential of $\mu(e)$. Then $V_\mu(M)\leqq 1$ on CE, but is not everywhere $\leqq 1$. For in that case we should have $\nu(F)=\mu(F)\leqq K_c(F)\leqq K_c(E)$.

The open set e_0 on which $V_\mu(M)>1+\eta$, where η is chosen >0 and so that $\nu(F)>K_c(E)(1+\eta)$, lies in E. It is composed of at most a denumerable infinity of domains D_i, and is not vacuous. In fact, there is at least one of these domains whose boundary contains a point of CE. For otherwise, by sweeping out from these domains successively, we should obtain a monotone-decreasing sequence of potentials, and a potential corresponding to the limiting function would everywhere, by §11, be $\leqq 1+\eta$. Let the corresponding distribution be $\nu'(e)$. Its total mass would remain $\mu(F)=\nu(F)$, since this quantity remains fixed during the weak convergence. Hence the distribution $\nu''(e)=\nu'(e)/(1+\eta)$ would lie on E and would have a potential everywhere $\leqq 1$; its total mass would therefore be $\leqq K_c(E)$. But the total mass is $\nu(F)/(1+\eta)>K_c(E)$.

Let D then be so chosen from the D_i that its boundary contains a point Q of CE. Then Q does not lie in F and $V_\mu(M)$ is continuous at Q. Consequently there is a neighborhood of Q in which everywhere $V_\mu(M)<1+\eta$, since $V_\mu(Q)\leqq 1$. But this neighborhood contains points of D. And this is a contradiction. Thus the proof is complete.

* C. de la Vallée Poussin, *Intégrales de Lebesgue*, Paris, 1916, p. 85.

We may digress at this point to indicate still another possible definition of capacity, and the value may be determined at once in terms of Maria's result,* that if a positive mass is distributed on a closed bounded set F the upper bound of its potential on F is at least as great as its upper bound on CF.

We define, in fact, $K_d(E)$ as *the upper bound of* $\mu(E)$, *where* $\mu(e)$ *is a distribution of positive mass on* E, *of which the potential* $V_\mu(M)$ *is* $\leqq 1$ *on* E.

Obviously $K_d(E) \geqq K_c(E)$. But also $K_d(E) \leqq K_c(E)$. In fact, given such a distribution $\mu(e)$, $V_\mu(M)$ is $\leqq 1$ on $\overline{E}$, the closed cover of E; for since $V_\mu(M)$ is lower semicontinuous the set where $V_\mu(M) \leqq 1$ is closed. Hence, by Maria's result, $V_\mu(M) \leqq 1$, everywhere.

Our results may be summarized in the equation

$$K_d(E) = K_c(E) = K_b(E) \leqq K_a(E) \leqq K(E), \tag{4}$$

where E is a bounded set measurable Borel, the equality signs being valid throughout if E is closed.

15.2. **Capable points.** A point Q is said to be a *capable* point of a bounded set E, measurable Borel, if no matter how small $\rho > 0$, the portion of E within a sphere of radius ρ and center Q is of positive capacity. The subset E' of incapable points is open with respect to E; that is, there is a neighborhood about an incapable point Q' of E which contains no points of E which are not points of E'. The set E_1 of capable points is therefore closed with respect to E. We shall have possibly different definitions of the subsets E', E_1 according as we use one definition or another of capacity.

Lemma. *If every point of a subset* E' *of a bounded set* E, *measurable Borel, is an incapable point* (*according to any of our definitions of capacity*), *then* $K_c(E') = 0$.

In fact, as de la Vallée Poussin remarks, in the memoir cited, each such point may be enclosed in a sphere of rational radius with center of rational coordinates, which contains no capable points; and there are only a denumerable infinity of such spheres.

If the set E is closed, the definition of capable point is independent of the choice among the definitions of capacity, and therefore the subsets E', E_1 are also. The set E_1 is likewise closed. It is called the *reduced* set. If the set E bears any distribution of mass for which the potential is bounded, the mass lies entirely on the reduced subset E_1. It does not follow that $K_a(E') = 0$ or $K(E') = 0$.

We return now for the rest of this section to the closed bounded set g,

* See §6, Remark III. But Maria's result depends on using Kellogg's Lemma, so that consideration of it in this paper would properly come after §18.

whose external frontier is s, and give a brief proof of Vasilesco's theorem:*

THEOREM. *If Q is a capable point of g and $\eta(M)$ is a conductor potential for g, then*

$$\limsup_{M=Q} \eta(M) = 1, \qquad \textit{for } M \textit{ in } W. \tag{5}$$

Let g_ρ be the closed cover of the portion of g within a sphere $\Gamma(\rho, Q)$ of center Q and radius ρ, and let Σ_ρ be the domain which is bounded by s_ρ, the external frontier of g_ρ. Part of the mass for the conductor potential $\eta(M)$ of g may lie on Σ_ρ; if so, we sweep it out, and obtain by Theorem II, §13, the conductor distribution on g_ρ, of total mass $K(g_\rho)$. We denote the conductor potential of g_ρ by $\eta_\rho(M)$.

The set g_ρ, by hypothesis, is of positive capacity. It follows that the upper bound of $\eta_\rho(M)$ is 1; for if the upper bound were $r<1$, the set g_ρ would sustain a mass of total amount $K(g_\rho)/r$, such that the upper bound of its potential would be $=1$. But throughout W, $\eta_\rho(M) \leqq 1$, and since it is not constant, we must have $\eta_\rho(M)<1$ for M in Σ_ρ; moreover $\eta_\rho(M) \leqq \eta(M)$. Hence

$$\text{u.b. } \eta(M) = 1, \qquad M \text{ in } \Gamma(\rho, Q).$$

And this proves the theorem.

We note that if g reduces to s, the boundary of Σ, and Q is a capable point of s, it follows from the lower semicontinuity of $\eta(M)$ that

$$\limsup_{P=Q} \eta(P) \leqq \limsup_{M=Q} \eta(M), \qquad P \text{ in } s,\ M \text{ in } \Sigma + B, \tag{6}$$

so that the second member of the inequality must have the value 1.

COROLLARY. *If g is of positive capacity there is at least one point of g where the conductor potential $\eta(M)$ for g has the value unity.*

In fact, the reduced set g_1 of g is not vacuous and has no isolated points, and in the neighborhood of any point P of g_1 there is a point Q of g_1, where, by Corollary II of §5.1, $\eta(M)$ is continuous. But then

$$\eta(Q) = \lim_{M=Q} \eta(M) = 1.$$

16. Points where a conductor potential has the value unity. We prove the following

THEOREM. *Let $\eta(M)$ be the conductor potential of s, as before, and let H be the subset of $G=s+B$ where $\eta(M)=1$. Then*

$$K_c(H) = K(G) = K(s).$$

* F. Vasilesco, *Sur les singularités des fonctions harmoniques*, Journal de Mathématiques, vol. 9 (1930), pp. 81–111; see p. 101.

In fact, $\eta(M)=\eta_0(M)$, $\eta_0(M)$ being the limiting function of the sweeping-out process, except for M on s. Hence $\eta(M)=1$ in B, if B is not vacuous. Let then t be the subset of s where $\eta(M)\leqq 1-\epsilon$, $1>\epsilon>0$. If t is not vacuous it is closed and bounded. We shall prove that $K_c(t)=0$.

Suppose that $K_c(t)$ is not zero. Let $\eta_t(M)$ be the potential obtained by sweeping out from the domain exterior to t the mass distribution of which $\eta(M)$ is the potential. Then $\eta_t(M)$ is the conductor potential for t, and $\eta_t(M)\leqq\eta(M)$. By the Corollary of §15, there is a point Q of t such that $\eta_t(Q)=1$. Hence $\eta(Q)=1$, which is a contradiction. Accordingly $K(t)=K_c(t)=0$.

The portion of s where $\eta(M)<1$ is the sum of a denumerable infinity of (overlapping) sets t, corresponding to decreasing values of ϵ, and therefore must have zero capacity K_c. Hence all of the mass of the conductor distribution must lie on H, and $K_c(H)=K(s)$, which was to be proved.

COROLLARY. *If $\eta(M)$ is a conductor potential for s, the conductor distribution lies entirely on that portion of s where $\eta(M)=1$.*

17. **Uniqueness of capacity potential.** We shall speak of a *capacity* distribution $\mu(e)$, for the moment, as any distribution of positive mass on G (G supposed to be of positive capacity), in total value equal to the capacity of s, provided that the upper bound of its potential $v(M)$ is less than or equal to unity. It cannot, in fact, be less than unity, from the definition of K_c, since $K_c(G)=K(G)=K(s)$. In particular, a conductor distribution for G is a capacity distribution. The following theorem was surmised by de la Vallée Poussin.*

THEOREM. *The potentials of all capacity distributions for G are identical in W; the capacity distributions are all identical on every set measurable Borel.*

We note first the following fact, which we may state as a lemma.

LEMMA I. *If E is a bounded set, measurable Borel, of positive spatial measure, $K_c(E)>0$.*

Let $m(e)$ be the measure of a Borel measurable set e, and define the mass distribution $\mu(e)$ by the equation

$$\mu(e)=m(e\cdot E).$$

The set function $\mu(e)$ is evidently additive and bounded, therefore completely additive, and represents a mass distribution on the bounded set E. Moreover,

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* de la Vallée Poussin, memoir cited, p. 232.

its potential is everywhere $\leqq 2\pi d^2$, where d is the diameter of E; hence $K_c(E) \geqq 1/(2\pi d^2) > 0$. Thus the lemma is proved.

Let $\eta(M)$ be the conductor potential for s, and $\nu(e)$ the corresponding distribution of positive mass, and let $\mu(e)$ be a capacity distribution and $v(M)$ its potential. We have immediately the following lemma.*

LEMMA II. *For M in Σ, $v(M) = \eta(M)$.*

In fact, if $\{\Sigma_n\}$ is the sequence of nested domains employed in forming $\eta(M)$, and $\{\eta_n(M)\}$ the corresponding sequence of potentials, we have $v(M) \leqq \eta_n(M)$, for all n, whence $v(M) \leqq \eta(M)$ in Σ. But then either $v(M) \equiv \eta(M)$, in Σ, or else $v(M) < \eta(M)$, M in Σ. The latter case is impossible, since it would follow, as in §2, that $\mu(G) < K(G)$.

LEMMA III. *The Dirichlet integral for a conductor potential is given by the equation*

$$D(\eta) = 4\pi K(s) = \int_\Sigma [\nabla\eta(M)]^2 dM. \tag{7}$$

Writing H for the subset of G where $\eta(M) = 1$, we have by §10 that $D(\eta)$ exists, whence

$$D(\eta) = 4\pi \int_W \eta(P) d\nu(e_P \cdot H) + 4\pi \int_W \eta(P) d\nu(e_P \cdot [W - H]).$$

But the second integral of the right-hand member is zero, since there is no mass on $W - H$, and the first integral, by (4) of §1, reduces to $4\pi \int_W 1 d\nu(e_P \cdot H) = 4\pi\nu(H)$. This establishes the first of equations (7). In order to establish the second result, it is sufficient to consider the case where G is of positive measure and perfect. The function $\eta(M)$ then has the value unity at almost all points of G, by Lemma I and the results of §16.

The partial derivative $\partial\eta/\partial x$ is measurable spatially in the Lebesgue sense, and the function $\eta(M)$ itself is absolutely continuous in x, by §3, on almost all lines parallel to the x-axis. On such lines the set $E(y, z)$ where $\eta(M) = 1$ is closed, and the total variation of $\eta(M)$ over $E(y, z)$ is 0. Hence $\partial\eta/\partial x = 0$ for almost all x on $E(y, z)$, and this, for almost all y, z. That is, $\partial\eta/\partial x = 0$ almost everywhere in G. Similar results hold for $\partial\eta/\partial y$ and $\partial\eta/\partial z$. Consequently $(\nabla\eta)^2 = 0$ almost everywhere on G, and

$$\int_W [\nabla\eta(M)]^2 dM = \int_\Sigma [\nabla\eta(M)]^2 dM,$$

which was to be proved.

* Ibid., p. 228.

LEMMA IV. *The quantities $D(v)$ and $D(\eta)$ are the same.*

We have

$$D(v) = 4\pi \int_W v(P) d\mu(e_P)$$

$$\leqq 4\pi \int_W 1 d\mu(e_P) = 4\pi\mu(G) = 4\pi K(s).$$

Hence $D(v) \leqq D(\eta)$. But also

$$D(v) = \int_W (\nabla v)^2 dM$$

$$\geqq \int_\Sigma (\nabla v)^2 dM = \int_\Sigma (\nabla \eta)^2 dM = D(\eta), \text{ by Lemmas II, III,}$$

so that $D(v) \geqq D(\eta)$. Hence $D(v) = D(\eta)$.

We can now complete the proof of the theorem by showing that $v(M)$ and $\eta(M)$ are everywhere the same; for it will then follow that the corresponding mass distributions are identical on all sets measurable Borel.

From Lemmas II, III, IV it is evident that $(\nabla v)^2 = 0$ almost everywhere on G, and thus, that the partial derivatives of $v - \eta$ are almost everywhere 0. But on almost all lines parallel to the x-axis the function $v - \eta$ is absolutely continuous in x and vanishes outside G, so that $v - \eta$ is zero almost everywhere.

Accordingly, for the spherical averages of §4,

$$v_\rho(M) = \eta_\rho(M), \qquad \text{for all } M,$$

and by (9′), §4,

$$v(M) = \lim_{\rho=0} v_\rho(M) = \lim_{\rho=0} \eta_\rho(M) = \eta(M), \qquad \text{for all } M.$$

This is what was to be proved.

18. **Short proof of Kellogg's Lemma.*** This lemma may be stated in the following form.

THEOREM. *If g is a bounded closed set of positive capacity, s its external frontier, and Σ the infinite region of boundary s, then s contains at least one point which is a regular boundary point of Σ.*

* O. D. Kellogg, loc. cit., p. 337. The author acknowledges indebtedness in connection with this proof to discussion with members of the seminar of 1934–35 at the Rice Institute, particularly with Dr. A. J. Maria. For abstract see Bulletin of the American Mathematical Society, vol. 40 (1934), p. 665. The same proof is given independently by F. Vasilesco, Comptes Rendus de l'Académie des Sciences, vol. 200 (1935), pp. 1173–1174.

Consider a conductor potential $\eta(M)$ for s.* Its mass lies entirely on s. Hence the reduced set for s may be taken as the perfect set F of §5.1. By Corollary II of §5.1 there is thus a capable point Q of s, such that $\eta(M)$ is continuous at Q. Consequently $\eta(Q)=1$, for lim sup $(M=Q)$ $\eta(M)=1$, by Vasilesco's theorem given in §15.2. But it is also a theorem of Vasilesco that if lim $(M=Q)$ $\eta(Q)=1$, for M in Σ, then Q is a regular point of s for Σ.†

19. **Second proof of Kellogg's Lemma, independent of Green's function.** The proof of Vasilesco's theorem, just cited, involves the result that a sufficient condition for a regular boundary point is the continuous vanishing of the Green's function at the point. A method of treatment, which perhaps is more direct, is based on Lebesgue's concept of *barrier*. A barrier for Σ at Q is a function $V(M, Q)$ which is continuous and superharmonic in Σ, which approaches zero at Q and has a positive lower bound in Σ outside any sphere with center Q. The construction of a barrier is immediate if the conductor potential at Q of the closed cover $s(\rho, Q)$ of the portion of s within a sphere $\Gamma(\rho, Q)$ has the value unity.‡

We find such a point Q by means of the following proposition.

LEMMA. *Let $\eta(\rho, M)$ be the conductor potential of $s(\rho, Q_1)$, $\eta(M)$ the conductor potential of s. If Q_1 is a capable point of s, there is a closed reduced set s_ρ, contained in $s(\rho, Q_1)$, of capacity as near that of $s(\rho, Q_1)$ as we please, such that*

$$\eta(\rho, P)=\eta(P)=1, \quad P \text{ in } s_\rho. \tag{8}$$

In fact, if we sweep out the mass of the conductor distribution from the domain which is exterior to $s(\rho, Q_1)$ we obtain the unique conductor distribution, for $s(\rho, Q_1)$. The set of capable points of $s(\rho, Q_1)$ where $\eta(\rho, P)=1$ bears all the mass of this conductor distribution, and therefore contains a closed subset s_ρ on which the total mass μ is as close to $K(s(\rho, Q_1))$ as we please. But $K(s_\rho)\geqq\mu$. Moreover $\eta(\rho, M)\leqq\eta(M)$, and so $\eta(M)=1$ on s_ρ also.

With the lemma thus proved, let Q_1 be a capable point of s, and construct the sets $s(\rho, Q_1)$, s_ρ with $\rho=\rho_1$. We note, in particular, that if the conductor potential of a set has the value unity at a point, that point must be a capable point. Next take a point Q_2 of s_{ρ_1}, which is a capable point of s_{ρ_1} and distant from Q_1 by less than ρ_1, and construct $s(\rho_2, Q_2)$ from s_{ρ_1} in the same way that

* Kellogg's Lemma depends on §5.1 and known results, and might have been inserted in that section. Hence "a" rather than "the" conductor potential. The theorem is put late in the present memoir in order to separate the theorems which involve it explicitly from those which do not.

† Vasilesco, loc. cit., p. 94. This theorem was wrongly cited as in Kellogg, loc. cit., at p. 331, by the author, in his paper *Application of Poincaré's sweeping-out process*, Proceedings of the National Academy of Sciences, vol. 19 (1933), pp. 457–461.

‡ O. D. Kellogg, loc. cit., pp. 227, 331.

$s(\rho_1, Q_1)$ is formed from s, taking $\rho_2 < \rho_1 - Q_1Q_2$. Similarly form $s(\rho_k, Q_k)$, s_{ρ_k}, from $s_{\rho_{k-1}}$, with the values ρ_k tending to zero. Of the closed sets s_{ρ_k}, each contains the next and none is empty; hence there is at least one point common to all of them, say Q. The conductor potentials $\eta(\rho_k, M)$ all have the value unity at Q.

Let now ρ be any value >0. The set $s(\rho, Q)$ contains the sets $s(\rho_k, Q_k)$ for k sufficiently large. Hence the conductor potential of $s(\rho, Q)$ dominates those of the sets $s(\rho_k, Q_k)$, since the latter may be obtained by sweeping out the former. Hence the conductor potential of $s(\rho, Q)$ has the value 1 at Q. This is what was to be proved.

VI. Applications

20. Necessary and sufficient condition for regular point. We prove the following

THEOREM. *A necessary and sufficient condition that Q be a regular point of s for Σ is that for every distribution of positive mass on a bounded set, the potential at Q be unchanged by the sweeping out of the portion of the mass in Σ.*

That the condition is sufficient is seen by the instance of the conductor potential, if Σ is an unbounded domain. If Σ is a bounded domain it is sufficient to consider the sweeping out of unit mass at a point M of Σ. From(15''), §14, if $g_n(P, M)$, $v_n(P, M)$ are respectively the Green's function and swept-out potential of unit mass at M, for Σ_n, $\{\Sigma_n\}$ being a sequence of nested regular domains for Σ, and if $g(P, M)$, $v(P, M)$ are the corresponding functions for Σ,

$$g_n(P, M) = \frac{1}{PM} - v_n(P, M),$$

$$g(P, M) = \frac{1}{PM} - v(P, M).$$

In fact, by definition, for P in Σ, $g(P, M) = \lim g_n(P, M)$; and the definition may be suitably extended to P in $C\Sigma$ by the above equation. But for P in Σ, $v(P, M) \leqq 1/(PM)$, so that if $v(Q, M) = 1/(QM)$ it follows that

$$\lim_{P=Q} v(P, M) = \frac{1}{QM}, \qquad \lim_{P=Q} g(P, M) = 0,$$

which is a sufficient condition for a regular point.*

* As is seen by means of a Kelvin transformation of the region into an infinite domain with bounded boundary. Or one may, with G. Bouligand (loc. cit), proceed directly from an analysis of the Green's function.

In order to prove the necessity of the condition, consider first the case where $U(M)$ is continuous on s and in its neighborhood. Then $V_0(M)$ is continuous at Q, being equal to $U(M)$ for M in $G=C\Sigma$, and taking on continuously the value $U(Q)$ as M tends to Q from Σ, as a property of the sequence solution at a regular point. But

$$(1) \qquad V(Q) = \lim_{\rho=0} V(\rho, Q) = \lim_{\rho=0} V_0(\rho, Q) = V_0(Q) = U(Q).$$

In the more general case, where $U(M)$ is not necessarily continuous or bounded on s, we may write, recalling the notation of §12.2,

$$V'^{(p)}(Q) = V_0'^{(p)}(Q) = U'^{(p)}(Q),$$

$$V'(Q) = \lim_{p=\infty} V'^{(p)}(Q) = \lim_{p=\infty} U'^{(p)}(Q) = U'(Q),$$

$$V(Q) = U''(Q) + V'(Q) = J''(Q) + U'(Q) = U(Q),$$

which was to be proved.

Our theorem may be summarized by the equation

$$(2) \qquad V(Q) = V_0(Q) = U(Q), \qquad Q \text{ a regular point of } s \text{ for } \Sigma,$$

since for all M, $V(M) \leqq V_0(M) \leqq U(M)$.

21. **The Dirichlet integral and the sweeping-out process.** The following theorem is a generalization of the statement that the value of the Dirichlet integral for the conductor potential is 4π times the capacity of the boundary set.

THEOREM. *Let $\{\Sigma_n\}$ be a sequence of nested regular domains for Σ, and $U(M)$ be a bounded potential of positive mass on a bounded set. If the sweeping-out process is carried out by means of the domains Σ_n, then the relation*

$$(3) \qquad D(V) = \lim_{n=\infty} D(V_n)$$

holds for the Dirichlet integrals.

LEMMA. *The theorem is true if $U(M)$ is continuous for M on s and in its neighborhood.*

In fact, the irregular boundary points of Σ are points where the conductor potential has a value <1, and therefore, by §16, form a subset of zero capacity K_e, and can sustain no portion of a mass distribution of which the potential is bounded. Hence if we denote by G_0 the set of points of $G=C\Sigma$, which are not irregular points of s, we shall have

$$D(V) = 4\pi \int_W V d\mu = 4\pi \int_W V(P) d\mu(G_0 \cdot e_P) + 4\pi \int_W V(P) d\mu(CG_0 \cdot e_P)$$

$$= 4\pi \int_W V(P) d\mu(G_0 \cdot e_P),$$

where $CG_0 = \Sigma + (CG_0) \cdot s$, so that $\mu(CG_0) = 0$. But by the theorem of §20, $V(P) = U(P)$ on $G_0 \cdot s$, and as a result of the sweeping-out process $V(P) = U(P)$ on $B = C(\Sigma + s)$, so that, by the relation (4) of §1,

$$D(V) = 4\pi \int_W U(P) d\mu(G_0 \cdot e_P) = 4\pi \int_W U(P) d\mu(e_P).$$

Now $\mu_n(e)$ converges weakly to $\mu(e)$ and $U(M)$ is continuous on s and in its neighborhood, whence

$$\int_W U(P) d\mu(e_P) = \lim_{n=\infty} \int_W U(P) d\mu_n(e_P).$$

But $U(P) = V_n(P)$ on $C\Sigma_n$, so that $\int U d\mu_n = \int V_n d\mu_n$ and finally

$$D(V) = \lim_{n=\infty} 4\pi \int_W V_n(P) d\mu_n(e_P),$$

which was to be proved.

Returning to the theorem, we may assume without loss of generality that the mass distribution lies entirely in Σ.

The quantities $D(V)$, $D(V_n)$ converge, since $V(M)$, $V_n(M)$ are bounded (see §10). Moreover, since $V(M) \leqq V_n(M)$ it follows by Corollary II of §10 that $D(V) \leqq D(V_n)$; consequently

$$D(V) \leqq \liminf_{n=\infty} D(V_n). \tag{4}$$

In order to obtain the complementary inequality, let Σ_δ be the portion of Σ distant from the boundary s by as much as δ, $\mu_{\delta n}(e)$ the distribution obtained by sweeping from Σ_n the portion of mass in Σ_δ, and $V_{\delta n}(M)$ the potential of the distribution $\mu_{\delta n}(e)$. Then by (9), §10,

$$D(V_n) - D(V_{\delta n}) = 4\pi \int_W V_n d\mu_n - 4\pi \int V_{\delta n} d\mu_{\delta n}$$

$$= 4\pi \int_W (V_n - V_{\delta n}) d\mu_{\delta n} + 4\pi \int_W V_n d(\mu_n - \mu_{\delta n}).$$

But the first integral, which is $D(V_n - V_{\delta n}, V_{\delta n}) = D(V_n, V_{\delta n}) - D(V_{\delta n})$, may also be written in the form $4\pi \int_W V_{\delta n} d(\mu_n - \mu_{\delta n})$, so that

$$D(V_n) - D(V_{\delta n}) = 4\pi \int_W (V_n + V_{\delta n}) d(\mu_n - \mu_{\delta n}).$$

Let n_δ be a value of n such that Σ_n contains Σ_δ. Then, for every $n > n_\delta$, $\mu_n(e) \geqq \mu_{\delta n}(e)$, by the process of §12.2. Hence N exists so that

$$0 \leqq D(V_n) - D(V_{\delta n}) < 8\pi N(\mu_n(\Sigma) - \mu_{\delta n}(\Sigma)),$$

and given $\epsilon > 0$ we can choose $\delta > 0$ so that

$$0 \leqq D(V_n) - D(V_{\delta n}) < \epsilon, \qquad n > n_\delta.$$

Let $V_\delta(M)$ be the potential obtained by sweeping out from Σ the portion of the original mass distribution in Σ_δ, according to the process of §12.2. By the lemma, we have

$$D(V_\delta) = \lim_{n=\infty} D(V_{\delta n}),$$

since $V_{\delta n}(M)$ is continuous on s and in its neighborhood. Hence

$$D(V) \geqq \lim_{n=\infty} D(V_{\delta n}) \geqq \limsup_{n=\infty} D(V_n) - \epsilon,$$

and

$$D(V) \geqq \limsup_{n=\infty} D(V_n). \tag{5}$$

From (4) and (5) we have (3), which is the statement to be proved. Incidentally, the inequality (4) shows that $D(V_n)$ is a decreasing function of n.

The theorem of this section is no longer true if the qualification "bounded" is removed from the hypothesis. In fact, if Σ is the domain exterior to a sphere and we are given a collection of point masses in Σ with limit point on the boundary s, such that the potential remains bounded on s, we shall have $D(U) = D(V_n) = \infty$, while $D(V)$ is finite.

22. **Condition that a function be a potential of positive mass.** We prove the following

THEOREM.* *Let $u(M)$ be harmonic in a domain Σ (with bounded boundary s), not identically zero, and, if Σ is an exterior domain, vanishing continuously at infinity. Let Σ' be a regular domain contained with its boundary s' in Σ, and let $V'(M)$ be the function constituted by the solutions of the Dirichlet problems (interior or exterior, as the case may be) for each of the domains comprising $B' = C(\Sigma' + s')$, with boundary values $u(M)$ on s'.*

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* Incidentally, this theorem provides an answer for G. Bouligand's Problem 2 (loc. cit., p. 16).

A necessary and sufficient condition that $u(M)$ be given for all M in Σ as a potential of some distribution of positive mass is that, for each Σ',

$$V'(M) \leqq u(M), \qquad M \text{ in } \Sigma - \Sigma'. \tag{6}$$

The mass may be distributed entirely on s.

If $u(M)$ is a potential of positive mass, the distribution lying accordingly on $C\Sigma$, it is superharmonic in each of the domains comprising $C(\Sigma'+s')$. Since the equation $V'(M)=u(M)$ is satisfied on each portion of s' which is the boundary of one of these domains, it follows that (6) is satisfied in the interior of the domain. Hence (6) is necessary.

In order to show that (6) is sufficient, consider a sequence $\{\Sigma_n\}$ of nested regular domains for Σ, and let $v_n(M)$ denote the corresponding functions $V'(M)$. We extend the definition of $v_n(M)$ by writing it equal to $u(M)$ in Σ_n. It is thus continuous in W. It possesses evidently the supermean property (see §2) for M in Σ_n and for M in $C(\Sigma_n+s_n)$. For points Q on s_n, we have, making use of (6),

$$v_n(Q) = u(Q) = A_u(\rho, Q) \geqq A_{v_n}(\rho, Q)$$

so that the supermean property holds there also. Hence $v_n(M)$ is superharmonic, and since it is not identically zero, is harmonic outside a bounded set and vanishes continuously at infinity, it is the potential of a positive distribution of mass. This mass is located entirely on s_n.

The functions $v_n(M)$ form a monotone-increasing sequence, their masses lie on sets which are bounded independently of n, and the limit function $v(M)=\lim v_n(M)$ is not identically infinite. In fact,

$$v_{n'}(M) \geqq v_n(M), \qquad \text{if } n' > n,$$

for $v_n(M)$ is harmonic in $C(\Sigma_n+s_n)$ and $v_{n'}(M)$ is superharmonic there, the two functions being identical in Σ_n+s_n. Moreover, the sets s_n are bounded, independently of n. Finally, the functions $v_n(M)$, forming an increasing sequence, are dominated by $u(M)$ in Σ, by hypothesis, and hence $v(M)$ is finite at every M in Σ.

It follows, by the theorem of §2.1, that the function $v(M)$ is a potential of positive mass, and since it is harmonic except on s the mass distribution must lie entirely on s. But, by construction, $v(M)$ is identical with $u(M)$ in Σ. This completes the proof.

23. **Sets of positive capacity.** Among other conditions, Wiener* gives the

* Wiener, loc. cit.; also Wiener, *The Dirichlet problem*, ibid., pp. 127–146. For a survey of this kind of problem and its extension to other special equations of elliptic type, see M. Brelot, *Le problème de Dirichlet sous sa forme moderne*, Mathematica, vol. 7 (1933), pp. 147–166.

following sufficient condition for the regularity of a point Q of s with respect to Σ. With our notation, G for the complement of Σ, $C(\rho, Q)$ for the spherical surface of center Q and radius ρ, and $\Gamma(\rho, Q)$ for the domain interior to $C(\rho, Q)$, it is expressed by the following statement:

The point Q is a regular point of s for Σ if there exists a sequence of values of r tending to zero and a constant $k>0$ such that the capacity of the set $G\cdot C(r, Q)$ is $\geqq kr$.

Likewise, it follows easily from the well known necessary and sufficient condition for a regular point, given by Wiener in the second of the memoirs just cited, that *Q is a regular point of s for Σ if*

$$K(G\cdot C(r, Q) + G\cdot \Gamma(r, Q)) \geqq kr$$

for a sequence of values of r tending to zero.

A point which satisfies this last condition may be called a *point of positive capacity density* in G. In particular, it follows from this capacity-density criterion that a point of s of positive spatial density in G is a regular boundary point with respect to Σ; and we have also the fact that if G is of positive capacity and contains a subset, similar to G, of diameter less than that of G, then it contains a point of positive capacity density, and its exterior frontier contains a regular boundary point for Σ. If it were true that every G of positive capacity contained a point of positive capacity density, we should have an independent proof of Kellogg's lemma.

In this section we content ourselves with proving the following theorem.

THEOREM. *Let g be a closed bounded set, g_0 its projection on any plane. If g_0 is of positive capacity (that is, with reference to Newtonian potential) then g is of positive capacity.*

The theorem will be proved if we can find a distribution of positive mass on g for which the potential is bounded. There exists such a distribution on g_0, by hypothesis; we represent it by $\mu^0(e)$. We take the plane of g_0 as the x, y plane.

Form a rectangular space net L, composed of a system of superimposed rectangular space lattices L_n, made by planes $x=\text{const.}$, $y=\text{const.}$, $z=\text{const.}$, the meshes of L_n being mutually distinct point sets of diameter $\leqq \delta_n$, where $\lim (n=\infty)\ \delta_n=0$. The projection of L_n on the x, y plane is a lattice L_n^0, and these lattices form a plane net L^0. To each mesh of L_n we let belong the faces of lowest algebraic values x, y, z respectively, and the single vertex of lowest algebraic values x, y, z. Thus L_n^0 is composed also of mutually distinct meshes.

Let $\omega_{i,n}^0$ be a mesh of L_n^0 which contains a point of g_0, and $\omega_{i,n}$ a mesh of L_n, of which $\omega_{i,n}^0$ is a projection, which contains a point of g; for definiteness, $\omega_{i,n}$ may be the one with least z-coordinate for its vertex. To the face $z=$const., of this mesh, of least z-coordinate, transfer the mass distribution $\mu^0(e\cdot\omega^0_{i,n})$, forming on this face a distribution $\mu_{i,n}(e)$. We write

$$\mu_n(e) = \sum_i \mu_{i,n}(e)$$

and thus obtain in space a bounded additive function of point sets measurable Borel.

There is a subsequence of these distributions $\mu_n(e)$ which converges in the weak sense to a distribution $\mu(e)$, and $\mu(e)$ lies entirely on g. In fact, if M is not on g, there will be a sphere of center M which contains no mesh $\omega_{i,n}$ for n sufficiently great. Without loss of generality we may restrict n to the sequence of the weak convergence.

Let M, P be points of space, Q, R their projections on the x, y plane, and write, with the notation of §1,

$$V^0(Q) = \lim_{N=\infty} \int_W h^N(Q, R)d\mu^0(e_R),$$

$$V(M) = \lim_{N=\infty} \int_W h^N(M, P)d\mu(e_P),$$

admitting the value $+\infty$, for the present, as a possible value of $V(M)$. Since $QR \leqq MP$, $h^N(M, P) \leqq h^N(Q, R)$, we note that

$$\int h^N(M, P)d\mu(e_P) \leqq \int h^N(Q, R)d\mu(e_P) = \lim_{n=\infty} \int h^N(Q, R)d\mu_n(e_P)$$

$$= \lim_{n=\infty} \sum_i \int h^N(Q, R)d\mu_{i,n}(e_P)$$

$$= \lim_{n=\infty} \sum_i \int h^N(Q, R)d\overset{0}{\mu}(e_R\cdot\overset{0}{\omega}_{i,n})$$

$$= \lim_{n=\infty} \int h^N(Q, R)d\mu^0(e_R).$$

Hence, since $\mu^0(e)$ does not involve n,

$$\int_W h^N(M, P)d\mu(e_P) \leqq \int_W h^N(Q, R)d\mu^0(e_R)$$

and

$$V(M) \leqq V^0(Q),$$

so that $V(M)$ is bounded for M in W. This is what was to be proved.

24. **Approximation on a closed set.*** Let g be a closed bounded set, and let the complement of g be written as an infinite domain Σ, plus possibly other domains $B_1, B_2, \cdots$. We speak of an exceptional point Q of g, as in §7.3, as a point of g such that in the neighborhood of Q there is contained in g a set of rectangles with sides parallel to arbitrary orthogonal directions x, y whose vertices constitute a set of positive spatial measure.

THEOREM. *If g contains no points which are exceptional, and $U(M)$ is given as superharmonic and continuous in a region with regular boundaries which encloses g in its interior, then there exists a sequence of functions $U_n(M)$, harmonic at all points of g, such that*

$$\lim_{n=\infty} U_n(M) = U(M), \qquad \textit{uniformly for all } M \textit{ in } g.$$

In any bounded subregion Ω contained strictly in the region mentioned in the theorem, $U(M)$ is the sum of a harmonic function and a potential of positive mass, bounded in total amount and distributed on Ω (Riesz's theorem, §4). This potential function may be taken as continuous in all space.

In fact, if we take a subregion Ω_0 contained strictly in Ω, the potentials due to the masses on Ω_0 and $\Omega-\Omega_0$ respectively are continuous in Ω; for, since each potential is lower semicontinuous, the sum cannot be continuous at a point unless both terms are also. Hence the potential due to the mass on Ω_0 is continuous throughout all space; and, since the potential of the mass on $\Omega-\Omega_0$ is harmonic in Ω_0, the desired resolution is obtained for the region Ω_0. There is no loss in generality in substituting Ω for Ω_0.

There is thus no loss in generality in assuming that $U(M)$ of the theorem is a potential of positive mass on a bounded set F, and is continuous throughout all space. For, having proved the theorem for the potential $U(M)$ we may add again the harmonic function to $U(M)$, $U_1(M)$, $U_2(M)$, $\cdots$ and thus obtain the original theorem.

The points of g may be enclosed in a finite number of spheres, and therefore in a finite number of regions with regular boundaries, constituting in

* J. L. Walsh, *The approximation of harmonic functions by harmonic polynomials and by harmonic rational functions*, Bulletin of the American Mathematical Society, vol. 35 (1929), pp. 499–544. Walsh's principal theorem for three dimensions is for bounded closed regions, such that every ray from some point of the interior contains a single boundary point (the boundary therefore is of spatial measure zero), assuming that the given function is continuous over the region and harmonic in the interior. See also C. T. Holmes, *The Approximation of Harmonic Functions in Three Dimensions by Harmonic Polynomials*, Dissertation, Harvard University, 1931, Theorems I and III.

Replacing a continuous function by a superharmonic one is a well known device. Likewise, a potential which is harmonic in a bounded open region Σ_0 can be approximated uniformly in any closed region contained in Σ_0 by a harmonic polynomial (see Walsh, loc. cit., p. 542.)

this way a finitely multiple open region, say g_1, with boundary s_1. Similarly we complete a sequence of finitely multiple open regions $g_1, g_2, \cdots$, with boundaries $s_1, s_2, \cdots, g_{n+1}$ to be contained strictly in g_n, and with lim $(n=\infty)$ $g_n=g$. We form the functions $U_n(M)$ as follows:

(i) $U_n(M)$ is to be a solution of the Dirichlet problem in the regions composing g_n, with boundary values $U(M)$,

(ii) $$U_n(M) = U(M) \text{ for } M \text{ in } Cg_n.$$

Then $U_n(M)$ is continuous in W and superharmonic; in fact, the supermean property is satisfied at every point. Since it is harmonic outside a bounded set, vanishing continuously with $U(M)$ at ∞, it is a potential of positive mass distributed on a bounded set. The functions $U_n(M)$ are dominated by $U(M)$, for all n, and form a monotone-increasing sequence with n; in fact, $U_{n+1}(M)$ is identical with $U_n(M)$ in Cg_n and $\geqq U_n(M)$ in g_n. Moreover none of the mass distribution for $U_n(M)$ lies outside a sufficiently large sphere, independent of n. Hence by the theorem of §2.1 the limit function

$$u(M) = \lim_{n=\infty} U_n(M)$$

is itself a potential of a distribution of positive mass on a bounded set.

We note that $u(M)$ is identical with $U(M)$. In fact, both functions are identical in Cg since every point of Cg is ultimately a point where $U_n(M)$ remains equal to $U(M)$ for all values of n sufficiently great. Moreover, by the theorem of §7.3, of which the proof applies when the set s is replaced by g, if Q is a point of g and M tends to Q from Cg, then

$$\begin{aligned} u(Q) &= \liminf_{M=Q} u(M) \\ &= \liminf_{M=Q} U(M) = U(Q), \end{aligned}$$

so that $u(Q)=U(Q)$. We have therefore

$$U(M) = \lim_{n=\infty} U(M), \qquad M \text{ in } W.$$

Since $U_n(M)$, $U(M)$ are continuous and since the sequence is monotone-increasing, the limit must be uniform on any bounded region. Moreover $U_n(M)$ is harmonic at all points of g. This completes the proof.

It is to be noted that any bounded closed set of spatial measure zero satisfies the conditions of the theorem: for example, a spherical surface with an isolated point in the interior, or a spherical surface supplemented with a Lebesgue spine, or a set consisting of a single point. The conditions that are given, however, are merely sufficient conditions. It is not presumed that the treatment of this problem is exhaustive, but merely that it shows an interesting application of the general methods.

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